Binomial Theorem

Note : $(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^2 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = (a+b)^3(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Pascal's Triangle : The cofficients of the expansion are arranged in an array. This array is called Pascal's Triangle.

The expansion of a binomial for any positive integral n

$$\boxed{(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots\ldots + {}^nC_{n-1} a.b^{n-1} + {}^nC_n b^n}$$

Observations :

1. The notation $\sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$ stands for ${}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots\ldots + {}^nC_n a^{n-n} b^n$, where $b^0 = 1 = a^{n-n}$
Hence the theorem can also be stated as $(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$.

2. The cofficients ${}^nC_n$ occuring in the binomial theorem are known as binomial cofficients.
3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e. one more than the index.
4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n-1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.
5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n+0 = n$ in the first term, $(n-1)+1 = n$ in the second term and so on $0+n = n$ in the last term. Thus it can be seen that the sum of the indices of a and b in every term of the expansion.

Some special cases

$a = x$ and $b = -y$ $\quad \boxed{(x-y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 - \ldots\ldots + (-1)^n {}^nC_n y^n}$

$a = 1$ and $b = x$ $\quad \boxed{(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots + {}^nC_n x^n}$

$a = 1$ and $b = -x$ $\quad \boxed{(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots\ldots + (-1)^n\, {}^nC_n x^n}$

General term : In General term of an expansion $(a+b)^n$ is $\boxed{T_{n+1} = {}^nC_n a^{n-n}.b^n}$ $[(n+1)^{th}$ term$]$

Middle terms

(i) If n is even, then the number of terms in the expansion will be $n+1$. Since n is even so $(n+1)$ is odd. Therefore, the middle term is $\left[\frac{n+1+1}{2}\right]^{th}$, i.e. $\left[\frac{n}{2}+1\right]^{th}$ term.

(ii) If n is odd, then $(n+1)$ is even, so there will be two middle terms in the expansion, namely, $\left[\frac{n+1}{2}\right]^{th}$ term and $\left[\frac{n+1}{2}+1\right]^{th}$ term.

(iii) In the expansion of $\left[x+\frac{1}{x}\right]^{2n}$, where $x \neq 0$, the middle term is $\left[\frac{2n+1+1}{2}\right]^{th}$ i.e. $(n+1)^{th}$ term, $2n$ is even. It is given by ${}^{2n}C_n x^n \left[\frac{1}{x}\right]^n = {}^{2n}C_n$ (constant).